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Supersymmetric model of spin-1/2 fermions on a chain

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Abstract

In a recent work, $N = 2$ supersymmetry has been proposed as a tool for the analysis of itinerant, correlated fermions on a lattice. In this paper, we extend these considerations to the case of lattice fermions with spin 1/2. We introduce a model for correlated spin-1/2 fermions with a manifest $N = 4$ supersymmetry, and analyse its properties. The supersymmetric ground states that we find represent holes in an anti-ferromagnetic background.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

For many condensed matter systems, the key to understanding the physical properties lies in the analysis of a quantum many body problem with strong correlations. For the analysis of such systems, approaches that go beyond the standard perturbative techniques are always needed. It has recently been proposed [1, 2] that, for a special class of lattice models for correlated fermions, supersymmetry can provide a tool for non-perturbative analysis. In these models, questions about the existence and degeneracies of strongly correlated ground states at zero energy are easily answered with the help of supersymmetry and elementary combinatorics. Explicit properties of these same ground states are being studied with techniques that are, in various ways, associated with supersymmetry [1, 3, 4].

In the formulation of supersymmetric lattice models, the starting point is the definition of two nilpotent fermionic operators denoted by $\overline{Q}$ and $Q = \overline{Q}^{\dagger}$. The Hamiltonian H is then built from these two Hermitian conjugate supercharges as $H = \{\overline{Q}, Q\}$ (this is often called $N = 2$ supersymmetry as there are two supersymmetry generators). This special algebraic structure implies a number of important properties, which lead to considerable computational simplification. In particular, the ground-state structure of these H can be analysed with the help of relatively simple combinatorics and of cohomology theory. It is important to emphasize that this approach remains well valid also in dimension $D > 1$ [5, 6] where, in general, very few non-perturbative techniques are available. The study of two-dimensional supersymmetric models in dimension $D > 1$ has revealed the existence of a large number of quantum systems characterized by a finite ground-state entropy at zero temperature, providing an intriguing new realization of the phenomenon of quantum frustration [5].

Further results have been obtained by studying a family of one-dimensional supersymmetric models, denoted with $M_k[\{x_a\}]$, which depend on $k-1$ free parameters x_a. These models M_k are described in terms of spinless fermions on a chain. The Hilbert space is restricted so that no more than k consecutive sites can be occupied. It was found [2] that there is a close connection between these models and a number of models that are well known in condensed matter theory: M_1 is closely related to the XXZ chain at $\Delta = -1/2$, $M_2[x = 0]$ connects to the $su(2|1)$-supersymmetric $t-J$ model, and the general model $M_k[\{x_a\}]$ can be related to a spin-$k/2$ XXZ model. These relations have permitted us to argue that, at some particular coupling $\{x_a\}$, the model M_k is described in the continuum limit by the kth minimal model of $N = 2$ superconformal field theory. With this result, all minimal universality classes of critical behaviour with $N = 2$ superconformal symmetry have been represented by lattice models with explicit $N = 2$ supersymmetry on a discrete, finite lattice.

Inspired by these results, and by the obvious wish to make a connection with lattice models for strongly correlated electrons, we have investigated possible generalizations to spin-1/2 fermions. Insisting on $SU(2)$ spin symmetry, one is quickly led to an algebraic structure having $N = 4$ rather than $N = 2$ supersymmetries. In this paper we follow this line of thought, and present an $N = 4$ supersymmetric lattice model for itinerant spin-1/2 fermions in one spatial dimension. For small lattices we find explicit supersymmetric ground states, showing that supersymmetry is unbroken.

Our construction is rather involved, and in its present form it is restricted to one spatial dimension. Nevertheless, our results do illustrate the potential use of $N = 4$ supersymmetry for the analysis of antiferromagnets that are doped with holes. As such they invite further analysis, in particular in the direction of models in $D = 2$ or higher spatial dimensions, where for $N = 2$ supersymmetric lattice models remarkable results have been obtained [5].

2. Lattice model with extended supersymmetry

In this main section, we present in a number of steps the construction of our $N = 4$ supersymmetric lattice model for spin-1/2 fermions. We begin by specifying how the spin $SU(2)$ algebra and supersymmetry give rise to $N = 4$ extended supersymmetry (section 2.1). We then specify a constrained Hilbert space for spin-1/2 fermions (section 2.2) and define the action of the four supercharges in section 2.3. Finally, in section 2.4, we present the Hamiltonian and the supersymmetric ground states that we obtained.

2.1. Algebraic structure

The first step is to fix the algebra formed by the supersymmetry and the $SU(2)$ generators. There are two couples of Hermitian conjugate supercharges, $\overline{Q}_\alpha$ and Q_α, that add or take out a fermion in the spin state α, with $\alpha = \uparrow, \downarrow$. These operators are nilpotent, i.e. they obey $(\overline{Q}_\alpha)^2 = (Q_\alpha)^2 = 0$. The generators of the spin symmetry, which we denote by J^a $(a = +, -, 0)$, can appear in the anti-commutation relations of the supercharges. In its simplest form, the so-called $SU(2)$-extended or $N = 4$ supersymmetry algebra reads

$$\{\overline{Q}_\alpha, \overline{Q}_\beta\} = \{Q_\alpha, Q_\beta\} = 0, \tag{1}$$

$$\{\overline{Q}_\alpha, Q_\beta\} = \delta_{\alpha,\beta} H + \gamma \sigma^a_{\alpha,\beta} J^a, \qquad \alpha, \beta = \uparrow, \downarrow, \quad a = +, -, 0, \tag{2}$$

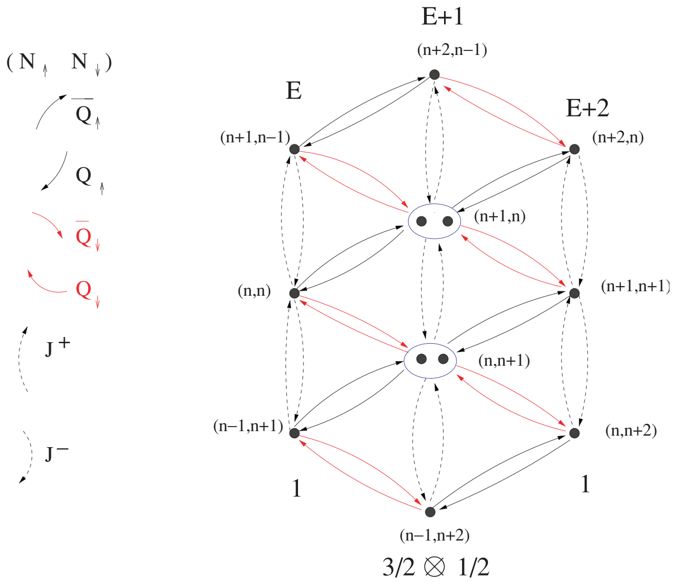

Figure 1. A characteristic supermultiplet forming a representation of the algebra (1), (2) for $\gamma = 1$. Each state (dot) in the supermultiplet is characterized by a pair $(N_\uparrow, N_\downarrow)$ which indicates the number of spin $\uparrow$ and $\downarrow$ fermions. The arrows show the action of the supersymmetry and $SU(2)$ generators on such states. This particular supermultiplet includes $SU(2)$ multiplets of spin 1/2, 1, 1 and 3/2.

where the $\sigma^a_{\alpha,\beta}$ are the Pauli matrices, γ is a constant and H will be identified as the Hamiltonian of the supersymmetric theory. One can verify, by using the Jacobi identities, that the above relations imply the following commutation relations

$$[J^a, \overline{Q}_\alpha] = \sigma^a_{\alpha,\beta}\overline{Q}_\beta, \qquad [J^a, J^b] = f^{abc}J^c, \tag{3}$$

where f^{abc} are the structure constants of the $SU(2)$ algebra which is thus included as a sub-algebra.

The energy operator

$$H = \{\overline{Q}_\uparrow, Q_\uparrow\} + \{\overline{Q}_\downarrow, Q_\downarrow\} \tag{4}$$

defined in (2) satisfies the relations

$$[H, J^a] = 0, \qquad [H, \overline{Q}_\alpha] = \gamma\overline{Q}_\alpha, \qquad [H, Q_\alpha] = -\gamma Q_\alpha. \tag{5}$$

From these relations, it follows that $H' = H - \hat{N}$, where $\hat{N}$ is the total fermion-number operator, commutes with all generators of the $N = 4$ supersymmetry algebra. This implies that H' will be constant on supermultiplets built from the generators of the algebra. A characteristic supermultiplet is shown in figure 1. It is composed of $SU(2)$-spin multiplets of spin 1/2, 1, 1 and 3/2. The Hamiltonian H is invariant under $SU(2)$-spin and it changes by $\pm\gamma$ under the action of one of the supercharges.

By fixing the normalization of the supercharges, we have three different algebras (2), which correspond to the cases $\gamma = 0$, $\gamma = -1$ and $\gamma = 1$.

When $\gamma = 0$, equations (1), (2) correspond to the usual Clifford algebra which is trivially satisfied by taking $\overline{Q}_\alpha = \sum_i c^+_{i,\alpha}$, where $c^+_{i,\alpha}$ creates a fermion at site i with spin α. The corresponding Hamiltonian is simply a constant, $H = L$, with L the number of lattice sites.

We have not found any other definition of the supercharges which respects equations (1), (2) for $\gamma = 0$.

If we restrict the Hilbert space by allowing at most one fermion at each site, we can verify that the supercharges $\overline{Q}_\alpha = \sum_i c^+_{i,\alpha}(1 - n_i)$, with n_i the fermion number operator at site i, and their conjugates form the algebra (1), (2) with $\gamma = -1$. The associated Hamiltonian is $H = 2L - \hat{N}$ and it is trivially diagonalizable.

Quite surprisingly, we have found a non-trivial realization of the algebra (1), (2) with $\gamma = 1$. This representation provides us with a model of interacting spin-1/2 fermions, which can be studied with the help of supersymmetry.

2.2. Defining the Hilbert space

To define the supercharges, we have first to fix the rules for determining the admissible configurations on a chosen lattice. Here we will consider the case of a system of spin-1/2 fermions on a one-dimensional chain.

The construction of an $N = 4$ supersymmetry structure for spin-1/2 fermions is far from evident. We found that a straightforward generalization of the realization of $N = 2$ supersymmetry for spin-less fermions, as presented in [1], fails to reproduce relation (2). To save this relation, we have to define both the Hilbert space and the supercharges in a quite evolved way.

The basic idea is to incorporate the $SU(2)$ structure at the level of the definition of the Hilbert space. We consider then a system of spin-1/2 fermions on a chain, subject to the following conditions:

- each site is occupied at most by one fermion,
- among the possible spin configurations associated with an even (odd) number of consecutive fermions, we allow only those which form a singlet (doublet) of $SU(2)$.

This last condition is illustrated explicitly with the help of some examples.

Consider first the case in which two fermions occupy neighbouring sites. The only permitted configuration is the one in which the two fermions are in a spin singlet state $(\uparrow\downarrow - \downarrow\uparrow)$. The other three possibilities $(\uparrow\uparrow, \downarrow\downarrow, \uparrow\downarrow + \downarrow\uparrow)$ are excluded.

Note that the fully polarized version of the model we are building corresponds exactly to the M_1 model, as fermions of the same spin cannot occupy sites that are nearest neighbour on the chain. The supercharges $\overline{Q}_\uparrow$ $(\overline{Q}_\downarrow)$ and $Q_\uparrow$ $(Q_\downarrow)$ act on the states with all the spins up (down) in the same way as the supercharge in the M_1 model

$$\overline{Q}_\uparrow(\ldots\circ\circ\uparrow\circ\circ\ldots) = \cdots + (\ldots\uparrow\circ\uparrow\circ\circ\ldots) - (\ldots\circ\circ\uparrow\circ\uparrow\ldots) + \cdots$$
$$Q_\uparrow(\ldots\circ\uparrow\circ\uparrow\circ\ldots) = \cdots + (\ldots\circ\circ\circ\uparrow\circ\ldots) - (\ldots\circ\uparrow\circ\circ\circ\ldots) + \cdots.$$

Here we represent an empty state with $\circ$.

Consider now the case of three consecutive fermions. From the $SU(2)$ representation theory, the direct product of three spin-1/2 representations decomposes into a spin-3/2 and two spin-1/2 irreducible representations: $1/2 \otimes 1/2 \otimes 1/2 = 3/2 \oplus 1/2 \oplus 1/2$. Among the $2^3 = 8$ possible configurations, we project out the states which participate in the spin-3/2 representation, keeping only the states which form a doublet $(\uparrow\uparrow\downarrow + \downarrow\uparrow\uparrow - 2\uparrow\downarrow\uparrow, \downarrow\downarrow\uparrow + \uparrow\downarrow\downarrow - 2\downarrow\uparrow\downarrow)$ and $(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow, \downarrow\downarrow\uparrow - \uparrow\downarrow\downarrow)$.

As we will show in the next section, the definition of the supercharges depends on the global $SU(2)$ properties of a number of consecutive fermions and not on the particular single spin configuration. In the following we therefore always suppose to select one singlet (doublet) state among the possible ones formed by $2n$ $(2n+1)$ consecutive fermions. Here we introduce

L=2

$N_\downarrow$ \ $N_\uparrow$	0	1
1	2	1
0	0	2

L=3

$N_\downarrow$ \ $N_\uparrow$	0	1
1	3	3
0	1	3

L=4

$N_\downarrow$ \ $N_\uparrow$	0	1	2
2	2	4	1
1	4	8	4
0	1	4	2

L=5

$N_\downarrow$ \ $N_\uparrow$	0	1	2
2	5	10	5
1	5	15	10
0	1	5	5

L=6

$N_\downarrow$ \ $N_\uparrow$	0	1	2	3
3	2	6	6	1
2	9	24	21	6
1	6	24	24	6
0	1	6	9	2

Figure 2. Table of the dimensions of the Hilbert space for each sector $(N_\uparrow, N_\downarrow)$ and for different chain lengths L. Here periodic boundary conditions are considered.

the following notation. We denote with $\ldots 2 \ldots$ the singlet formed by two consecutive fermions $(\ldots(\uparrow\downarrow - \downarrow\uparrow)\ldots)$, with $\ldots 3_\alpha \ldots, \alpha = \uparrow, \downarrow$, the two states of a doublet formed by three consecutive fermions, etc. In the case of a cluster of three fermions, this notation will represent one of the two possible doublets, which can be chosen by following a certain criterion. For example, $\ldots 3_\uparrow \ldots$ can indicate either the state $\ldots(\uparrow\uparrow\downarrow + \downarrow\uparrow\uparrow - 2\uparrow\downarrow\uparrow)\ldots$ or the state $\ldots(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)\ldots$. Starting from physical considerations, a possible criterion for choosing a particular singlet or doublet will be discussed later.

In general, denoting with $\ldots 2n \ldots$ and $\ldots(2n+1)_\alpha \ldots$ a cluster of $2n$ and $2n+1$ consecutive fermions, we require that:

$$J^a 2n = 0, \qquad J^a(2n+1)_\alpha = \sigma^a_{\alpha\beta}(2n+1)_\beta, \tag{6}$$

where $a = +, -, 0$, $\alpha = \uparrow, \downarrow$ and $\sigma^a_{\alpha\beta}$ are the Pauli matrices. A typical configuration reads

$$\ldots \circ\circ\circ 2 \circ 3_\uparrow \circ 3_\downarrow \circ \uparrow \circ\circ \uparrow \circ 4 \circ\circ \ldots \tag{7}$$

This defines an Hilbert space $\mathcal{H}$ with subspaces $\mathcal{H}_{N_\uparrow,N_\downarrow}$ defined by the condition that the eigenvalue of the spin α fermion number operator $\hat{N}_\alpha$ be equal to N_α. In figure 2 we give the dimension of the Hilbert space (i.e. all the different configurations of clusters on the chain) for a chain of length $L = 2, 3, \ldots, 6$, when periodic boundary conditions are taken.

2.3. The supersymmetry generators

Instead of writing the supersymmetry operators in the second quantized form, we prefer to list the non-vanishing amplitudes they entail in terms of fermion clusters, defined in the previous section. Before giving the complete construction of the supercharge operators, we illustrate their action on some explicit examples shown in figure 3.

Consider two clusters of types $3_\downarrow$ and $5_\uparrow$ separated by one empty site,$\ldots 3_\downarrow \circ 5_\uparrow \ldots$. The operator $\overline{Q}_\uparrow$ will fill the empty space between the two clusters with a fermion with spin $\uparrow$ and will create, with a coefficient defined below, an unique cluster of type $9_\uparrow$, i.e., $\overline{Q}_\uparrow \ldots 3_\downarrow \circ 5_\uparrow \ldots \to \ldots 9_\uparrow \ldots$.

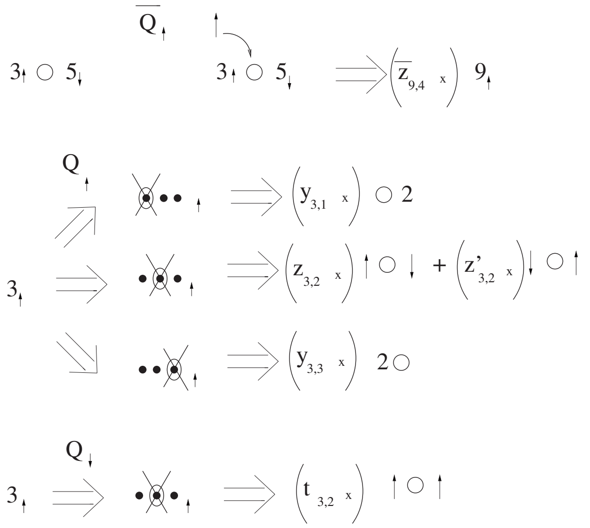

Figure 3. Examples of the action of the supersymmetry generators (with relative amplitudes) on the restricted Hilbert space.

Suppose now that two consecutive clusters have the same spin, as in the configuration $\ldots(2n+1)_\uparrow \circ (2m+1)_\uparrow \ldots$: the constraints on the Hilbert space imply that only a fermion with spin $\downarrow$ can be added, thus giving the possible amplitudes $\overline{Q}_\uparrow \ldots (2n+1)_\uparrow \circ (2m+1)_\uparrow \ldots \to 0$ and $\overline{Q}_\downarrow \ldots (2n+1)_\uparrow \circ (2m+1)_\uparrow \ldots \to \ldots (2n+2m+3)_\uparrow \ldots$.

The operators Q_α act in the opposite sense. Given a cluster of size n, they take out a fermion with spin α at each position along the cluster, always respecting the conditions on the Hilbert space. Consider for example the clusters $3_\uparrow$ and $3_\downarrow$. By applying the operator $Q_\uparrow$, we find the states $Q_\uparrow 3_\uparrow \to \circ 2, \downarrow \circ \uparrow, \uparrow \circ \downarrow, 2 \circ$ and $Q_\uparrow 3_\downarrow \to \downarrow \circ \downarrow$.

We can now introduce in a compact form the amplitudes defining the supersymmetry operators $\overline{Q}_\alpha$

$$
\begin{aligned}
&\overline{Q}_\gamma \ldots (2n+1)_\alpha \circ (2m+1)_\beta \ldots \\
&\qquad = (1-\delta_{\alpha,\beta})[\delta_{\gamma,\alpha}\bar{z}_{(2n+2m+3,2n+2),\gamma} + \delta_{\gamma,\beta}\bar{z}'_{(2n+2m+3,2n+2),\gamma}] \ldots (2n+2m+3)_\gamma \ldots \\
&\qquad\quad + \delta_{\alpha,\beta}(1-\delta_{\alpha,\gamma})\bar{t}_{(2n+2m+3,2n+2),\gamma} \ldots (2n+2m+3)_\alpha \ldots, \\
&\overline{Q}_\gamma \ldots 2n \circ 2m \ldots \\
&\qquad = \bar{y}_{(2n+2m+1,2n+1),\gamma} \ldots (2n+2m+1)_\gamma \ldots, \\
&\overline{Q}_\gamma \ldots 2n \circ (2m+1)_\alpha \ldots \\
&\qquad = (1-\delta_{\gamma,\alpha})\bar{x}_{(2n+2m+2,2n+1),\gamma} \ldots (2n+2m+2) \ldots, \\
&\overline{Q}_\gamma \ldots (2n+1)_\alpha \circ 2m \ldots \\
&\qquad = (1-\delta_{\gamma,\alpha})\bar{x}_{(2n+2m+2,2n+2),\gamma} \ldots (2n+2m+2) \ldots,
\end{aligned}
\tag{8}
$$

and Q_α

$$
\begin{aligned}
&Q_\gamma \ldots (2n+1)_\alpha \ldots \\
&\qquad = \delta_{\gamma,\alpha} \sum_{i=1,3,\ldots}^{2n+1} y_{(2n+1,i),\gamma} \ldots (2n+1-i) \circ (i-1) \ldots
\end{aligned}
$$

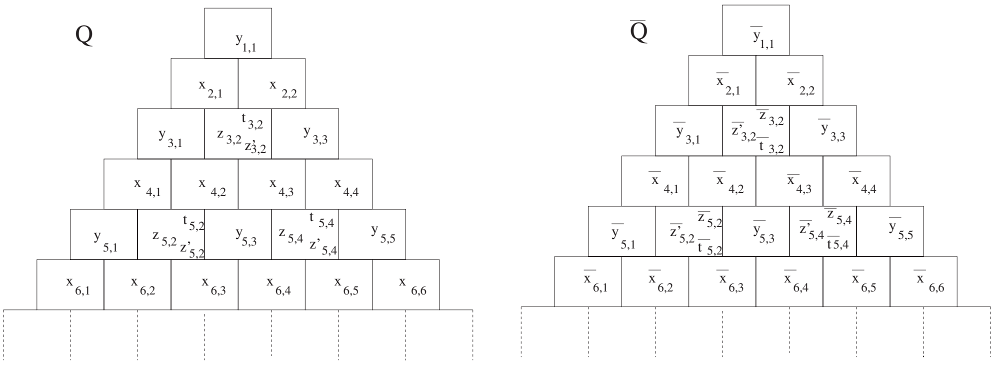

Figure 4. The amplitudes defining the supersymmetry generators can be listed in a pyramidal structure.

$$
+\delta_{\gamma,\alpha}(1-\delta_{\gamma,\beta})\sum_{i=2,4,\ldots}^{2n} z_{(2n+1,i),\gamma}\ldots(2n+1-i)_\alpha\circ(i-1)_\beta\ldots
$$
$$
+\delta_{\gamma,\alpha}(1-\delta_{\gamma,\beta})\sum_{i=2,4,\ldots}^{2n} z'_{(2n+1,i),\gamma}\ldots(2n+1-i)_\beta\circ(i-1)_\alpha\ldots
$$
$$
+(1-\delta_{\gamma,\alpha})\sum_{i=2,4,\ldots}^{2n} t_{(2n+1,i),\gamma}\ldots(2n+1-i)_\alpha\circ(i-1)_\alpha\ldots,
$$

$$
\begin{aligned}
&Q_\gamma\ldots 2n\ldots\\
&\quad=(1-\delta_{\gamma,\alpha})\left[\sum_{i=1,3,\ldots}^{2n-1} x_{(2n,i),\gamma}\ldots(2n-i)_\alpha\circ(i-1)\ldots\right.\\
&\qquad\left.+\sum_{i=2,4,\ldots}^{2n} x_{(2n,i),\gamma}\ldots(2n-i)\circ(i-1)_\alpha\ldots\right].
\end{aligned}
\tag{9}
$$

In figure 3 we show some of these matrix elements. The whole set of coefficients defining the supersymmetry generators can be displayed as in figure 4.

The values of these coefficients are determined by imposing the algebra (1), (2). The procedure goes as follows. We consider, for example, the operators $\overline{Q}_\uparrow$ and $Q_\downarrow$ and we determine the coefficients in figure 4 row by row starting from the top of the pyramid. The conditions of nil-potency of the supersymmetry operators, $\overline{Q}_\uparrow^2=0$ and $Q_\downarrow^2=0$, lead to a series of recursions of the type:

$$
\begin{aligned}
X_{(n,m),\alpha}X_{(q,n+1),\alpha}&=X_{(q-m,n+1-m),\alpha}X_{(q,m),\alpha}\\
\bar{X}_{(n,m),\alpha}\bar{X}_{(q,n+1),\alpha}&=\bar{X}_{(q-m,n+1-m),\alpha}\bar{X}_{(q,m),\alpha},
\end{aligned}
\tag{10}
$$

where $X_{(a,b),\alpha}$ denotes one of the possible types of amplitude and $m\leqslant n+1\leqslant q$. The above recursions leave some free parameters which are then tuned to satisfy the further relation $\{\overline{Q}_\uparrow,Q_\downarrow\}=-J^+$. Modulo an overall normalization of the supercharges, $\overline{Q}_\uparrow\to N\times\overline{Q}_\uparrow$ and $Q_\downarrow\to N^{-1}\times Q_\downarrow$ (we will show later that the norm of the cluster states $2n$ and $(2n+1)_\alpha$ depends on the choice of this normalization), all the possible parameters are then determined.

The values found in this way are shown in figure 5, where a particular normalization has been chosen. It is important to remark that, already at this stage, the number of constraints

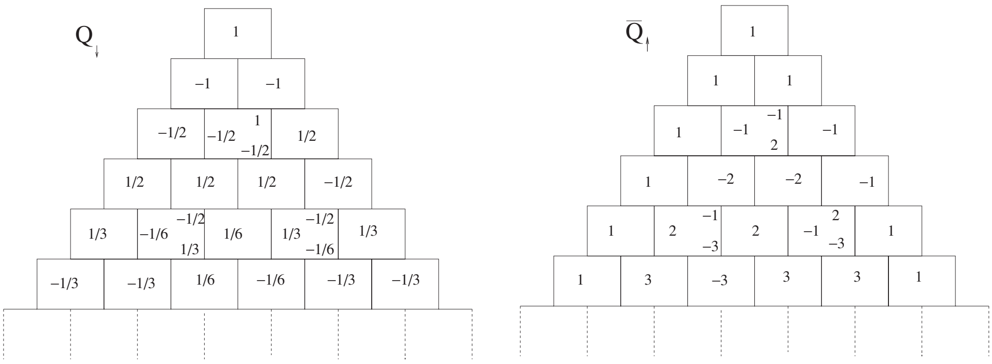

Figure 5. Values of the amplitudes of the supersymmetry generators fixed by imposing the algebra (1), (2).

imposed by the algebra is larger than the number of adjustable parameters $\bar{X}_{a,b}$. This is the reason why, for example, we cannot adjust the parameters to realize the algebra (2) with $\gamma = -1$ or $\gamma = 0$.

The next step is to find the coefficients of the other two operators, $\overline{Q}_\downarrow$ and $Q_\uparrow$. The conditions $\{\overline{Q}_\uparrow, \overline{Q}_\downarrow\} = 0$ and $\{Q_\uparrow, Q_\downarrow\} = 0$ (see (1)) yield the relations between the up and down sectors

$$
\begin{aligned}
&\bar{X}_{(2n+1,2m),\uparrow} = \bar{X}_{(2n+1,2m),\downarrow}, && X_{(2n+1,2m),\uparrow} = X_{(2n+1,2m),\downarrow} \quad \text{for} \quad X = t, z, z', \\
&\bar{y}_{(2n+1,2m+1),\uparrow} = (-1)^n \bar{y}_{(2n+1,2m+1),\downarrow}, && y_{(2n+1,2m+1),\uparrow} = (-1)^n y_{(2n+1,2m+1),\downarrow}, \\
&\bar{x}_{(2n,2m+1),\uparrow} = (-1)^{n+m} \bar{x}_{(2n,2m+1),\downarrow}, && \bar{x}_{(2n,2m),\uparrow} = (-1)^m \bar{x}_{(2n,2m),\downarrow}, \\
&x_{(2n,2m+1),\uparrow} = (-1)^{n+m} x_{(2n,2m+1),\downarrow}, && x_{(2n,2m),\uparrow} = (-1)^m x_{(2n,2m),\downarrow}.
\end{aligned}
\tag{11}
$$

The above sign rules can be nicely explained in terms of the parity of the singlets under a spin reversal. In particular, if a string of $2n$ fermions takes a sign $(-1)^n$ under a spin reversal operation, then equations (11) are easily obtained. Note that a singlet formed by two fermions, $\uparrow\downarrow - \downarrow\uparrow$ has odd parity, and the two singlets of four consecutive fermions, $\uparrow\uparrow\downarrow\downarrow + \uparrow\downarrow\downarrow\uparrow + \downarrow\downarrow\uparrow\uparrow + \downarrow\uparrow\uparrow\downarrow - 2\,\uparrow\downarrow\uparrow\downarrow - 2\,\downarrow\uparrow\downarrow\uparrow$ and $\uparrow\uparrow\downarrow\downarrow - \uparrow\downarrow\downarrow\uparrow + \downarrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\downarrow$, have even parity.

Choosing a normalization such that $\bar{y}_{(2n+1,1),\uparrow} = \bar{x}_{(2n,1),\uparrow} = 1$, the general solution for the coefficients reads as follows:

$$
\overline{Q}_\uparrow \begin{cases}
\bar{t}_{(2n+1,2m),\uparrow} = (-1)^{mn+1} \binom{n+1}{m}, & \bar{y}_{(2n+1,2m+1),\uparrow} = (-1)^{nm} \binom{n}{m}, \\
\bar{z}_{(2n+1,2m),\uparrow} = (-1)^{(m-1)(n-1)+1} \binom{n}{m-1}, & \bar{z}'_{(2n+1,2m),\uparrow} = (-1)^{m(n-1)+1} \binom{n}{m}, \\
\bar{x}_{(2n,2m),\uparrow} = (-1)^{m(n-1)} \binom{n}{m}, & \bar{x}_{(2n,2m+1),\uparrow} = (-1)^{mn} \binom{n}{m},
\end{cases}
$$

$$
Q_\uparrow \begin{cases}
t_{(2n+1,2m),\uparrow} = \frac{(-1)^{mn+1}}{m} \binom{n}{m}^{-1}, & y_{(2n+1,2m+1),\uparrow} = \frac{(-1)^{nm}}{m+1} \binom{n+1}{m+1}^{-1}, \\
z_{(2n+1,2m),\uparrow} = \frac{(-1)^{(m-1)(n-1)+1}}{m+1} \binom{n+1}{m+1}^{-1}, & z'_{(2n+1,2m),\uparrow} = \frac{(-1)^{m(n-1)+1}}{m} \binom{n+1}{m}^{-1}, \\
x_{(2n,2m),\uparrow} = \frac{(-1)^{m(n-1)}}{m} \binom{n}{m}^{-1}, & x_{(2n,2m+1),\uparrow} = \frac{(-1)^{mn}}{m+1} \binom{n}{m+1}^{-1}.
\end{cases}
$$

From the above formulae, the amplitudes associated with $\overline{Q}_\downarrow$ and $Q_\downarrow$ are easily obtained by using relations (11).

The norms $N_{2n} \doteq \langle 2n|2n\rangle$ and $N_{2n+1} \doteq \langle (2n+1)_\alpha|(2n+1)_\alpha\rangle$ of the cluster states $2n$ and $(2n+1)$ are fixed by using the fact the operators $\overline{Q}_\alpha$ and Q_α are self-conjugate. For example, the condition

$$\langle 2n|\overline{Q}_\uparrow|(2n-2q-1)_\downarrow \circ 2q\rangle = \langle (2n-2q-1)_\downarrow \circ 2q|Q_\uparrow|2n\rangle \tag{12}$$

yields the following relation between the norms and the amplitudes associated with the supersymmetry generators

$$\bar{x}_{(2n,2n-2q),\uparrow} N_{2n} = x_{(2n,2n-2q),\uparrow} N_{2n-2q-1} N_{2q}. \tag{13}$$

Given as initial conditions $N_0 = N_1 = 1$, one finds that the above condition is satisfied if

$$N_{2n} = \frac{1}{(n!)^2} \qquad \text{and} \qquad N_{2n+1} = \frac{1}{(n+1)!n!}. \tag{14}$$

One can verify that all the other self-conjugacy relations like equation (13) are all respected by taking the norms (14), thus confirming the consistency of all the construction. Normalizing the basis states as $|2n\rangle \to N_{2n}^{-1/2}|2n\rangle$ and $|(2n+1)_\alpha\rangle \to N_{2n+1}^{-1/2}|(2n+1)_\alpha\rangle$, the amplitudes of the supercharge operators take a very simple form:

$$\begin{aligned}
\bar{t}_{(2n+1,2m),\uparrow} &= t_{(2n+1,2m),\uparrow} = (-1)^{mn+1}\sqrt{\frac{(n+1)(n-m)}{m}}, \\
\bar{y}_{(2n+1,2m+1),\uparrow} &= y_{(2n+1,2m+1),\uparrow} = (-1)^{nm}\sqrt{\frac{1}{n+1}}, \\
\bar{z}'_{(2n+1,2m),\uparrow} &= z'_{(2n+1,2m),\uparrow} = (-1)^{m(n-1)+1}\sqrt{\frac{m}{(n+1)(n-m+1)}}, \\
\bar{z}_{(2n+1,2m),\uparrow} &= z_{(2n+1,2m),\uparrow} = (-1)^{(m-1)(n-1)+1}\sqrt{\frac{n-m+1}{m(n+1)}}, \\
\bar{x}_{(2n,2m),\uparrow} &= x_{(2n,2m),\uparrow} = (-1)^{m(n-1)}\sqrt{\frac{1}{m}}, \\
\bar{x}_{(2n,2m+1),\uparrow} &= x_{(2n,2m+1),\uparrow} = (-1)^{mn}\sqrt{\frac{1}{n-m}}.
\end{aligned} \tag{15}$$

With this last normalization, the operators $\overline{Q}_\alpha$ and Q_α are conjugate w.r.t. one another. In the examples presented in section 2.4, we shall denote normalized basis states with brackets, $\langle\cdots\rangle$.

Having constructed completely the supersymmetry operators, one can verify that they realize the algebra (1), (2). We have thus obtained a nonlinear realization of the $N = 4$ supersymmetry algebra (1), (2) at $\gamma = 1$, on a Hilbert space built from spin-1/2 fermions. The corresponding Hamiltonian and supersymmetric ground states are discussed in the next section.

2.4. Hamiltonian and supersymmetric ground states

The Hamiltonian (4) provided by the supercharges (8) and (9) acts on the Hilbert space $\mathcal{H}$ composed by clusters of electrons forming singlets or doublets of total spin, all spaced by one or more unoccupied sites. The amplitudes associated with the terms of the Hamiltonian listed below are given in the appendix. The action of the Hamiltonian can be divided into the following:

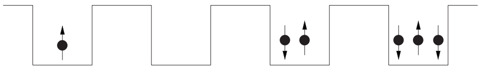

Figure 6. Spin-1/2 particles confined in a one-dimensional periodic potential.

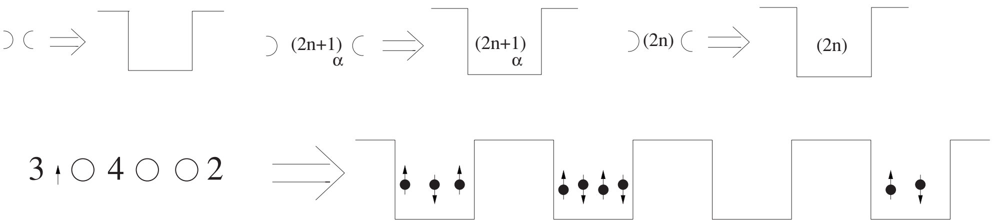

Figure 7. The model of spin-1/2 fermions on a chain is mapped into a model of spin-1/2 fermions confined to a one-dimensional periodic potential.

(1) *Split–join terms*. Two clusters spaced by one site split and join to form all the possible pairs of clusters that preserve the number of fermions with spin ↑ and ↓ and respect the constraints on the Hilbert space:

- $2n \circ 2m \leftrightarrow (1-\delta_{\alpha,\beta})(2n+2m-2q-1)_\alpha \circ (2q+1)_\beta$ with $\alpha, \beta = \uparrow, \downarrow$ and $q = 0, 1, \ldots, n+m-1$.
- $2n \circ 2m \leftrightarrow (2n+2m-2q) \circ 2q$ with $q = 0, 1, \ldots, m-1$,
- $2n \circ (2m+1)_\alpha \leftrightarrow (2n+2m-2q+1)_\alpha \circ 2q$ with $q = 0, 1, \ldots, n+m$,
- $2n \circ (2m+1)_\alpha \leftrightarrow (2n+2m-2q) \circ (2q+1)_\alpha$ with $q = 0, \ldots, n+m-1$.

Among these amplitudes we find the hopping terms where a single cluster is moved by one position: $2n\circ \leftrightarrow 2n\circ$ and $(2n+1)_\alpha\circ \leftrightarrow (2n+1)_\alpha\circ$.

(2) *Potential terms (diagonal terms)*:

- $2n \leftrightarrow 2n$,
- $(2n+1)_\alpha \leftrightarrow (2n+1)_\alpha$.

Some explicit examples are given below:

$$\begin{aligned} H\langle\circ\uparrow\circ\downarrow\circ\rangle &= 5\langle\circ\uparrow\circ\downarrow\circ\rangle + 2[\langle\uparrow\circ\circ\downarrow\circ\rangle + \langle\circ\uparrow\circ\circ\downarrow\rangle] \\ &\quad + \langle\circ 2\circ\circ\rangle + \langle\circ\circ 2\circ\rangle + \langle\circ\downarrow\circ\uparrow\circ\rangle, \\ H\langle\circ 4\circ\rangle &= \tfrac{22}{3}\langle\circ 4\circ\rangle + \tfrac{2\sqrt{2}}{3}[\langle\circ 3_\uparrow\circ\downarrow\rangle + \langle\circ 3_\downarrow\circ\uparrow\rangle] - \tfrac{\sqrt{2}}{3}[\langle 3_\uparrow\circ\downarrow\circ\rangle + \langle 3_\downarrow\circ\uparrow\circ\rangle] \\ &\quad - \tfrac{\sqrt{2}}{3}[\langle\circ\uparrow\circ 3_\downarrow\rangle + \langle\circ\downarrow\circ 3_\uparrow\rangle] + \tfrac{2\sqrt{2}}{3}[\langle\uparrow\circ 3_\downarrow\circ\rangle + \langle\downarrow\circ 3_\uparrow\circ\rangle] \\ &\quad - \tfrac{1}{3}[\langle 4\circ\circ\rangle + \langle\circ\circ 4\rangle] - \tfrac{4}{3}[\langle\circ 2\circ 2\rangle + \langle 2\circ 2\circ\rangle]. \end{aligned} \tag{16}$$

A possible interpretation of our supersymmetric model is the following. Consider a system of spin-1/2 particles subjected to a one dimensional periodic potential as shown in figure 6. In the mapping given in figure 7, a cluster of $2n$ $(2n+1)$ fermions represent a number $2n$ $(2n+1)$ of spin-1/2 particles confined in a well and forming a singlet (doublet) state of total spin. This state could originate from an anti-ferromagnetic interaction between these particles. We can, for example, suppose that the particles confined in a well form the ground state of an anti-ferromagnetic Heisenberg Hamiltonian. Actually, this hypothesis would provide us with a reasonable criterion to select one singlet (doublet) among the possible ones formed by $2n$ $(2n+1)$ spin-1/2 particles. We could imagine this system has two characteristic times, τ_1, τ_2, with $\tau_1 \ll \tau_2$. In a time of order of τ_1, the particles arrange themselves to form the ground state of an anti-ferromagnetic spin energy operator. From this point of view, the Hamiltonian

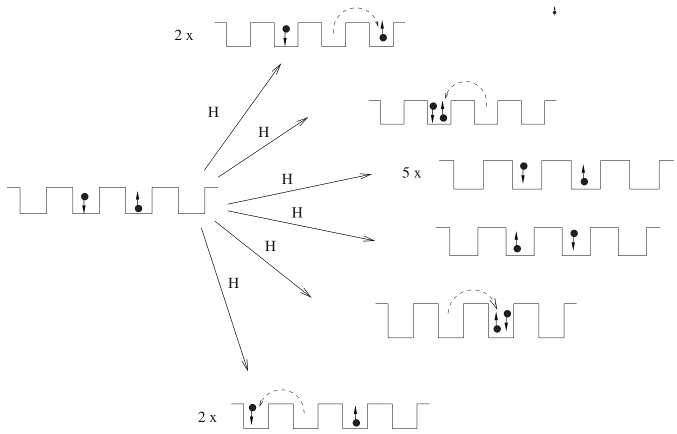

Figure 8. Transfer of particles between adjacent wells as induced by the supersymmetric Hamiltonian.

we have defined describes a particular dynamics induced by the transfer of particles between adjacent wells (see figure 8), which takes place over times of the order of τ_2.

We consider now the case of a chain of finite length L. It is important to remember here that to realize the algebra (1), (2) with $\gamma = 1$ we need to allow for strings of consecutive fermions of any size. The construction of the supersymmetry generators as given in the previous section is thus consistent only in the case of an infinite chain. If the chain is finite, specific boundary conditions tend to be in conflict with the supersymmetry algebra in the subspaces $\mathcal{H}_{N_\uparrow,N_\downarrow}$ with $N_\uparrow + N_\downarrow \geqslant L - 1$, as the notions of empty sides on the left and right of a length $(L-1)$ string become ambiguous.

The first example we consider is the case $L = 4$, with periodic boundary conditions. The corresponding dimensions of the Hilbert spaces $\mathcal{H}_{N_\uparrow,N_\downarrow}$ are given in figure 2. We have computed the eigenvalues of the Hamiltonian H in each sector $\mathcal{H}_{N_\uparrow,N_\downarrow}$ with $N_\uparrow + N_\downarrow < 3$, thus avoiding the sectors (1, 2), (2, 1) and (2, 2) where, due to the periodic boundary conditions, the action of the supercharges becomes ambiguous. We have also kept track of the multiplet structure under the supersymmetry algebra, and of the eigenvalues t of the action of the translation operator (satisfying, in general, $t^L = 1$). The complete results are given in figure 9. To complete the multiplets at $t = -1, i, -i$ that involve states at fermion numbers (2, 1) and (1, 2), we have assumed that the supercharges $\overline{Q}_\uparrow$ and $\overline{Q}_\downarrow$ annihilate these states.

It is important to stress the existence of a unique ground state, $|gs\rangle_4$ with energy $E = 0$, and thus $\overline{Q}_\alpha|gs\rangle_4 = Q_\alpha|gs\rangle_4 = 0$, in the sector (1, 1). Its explicit form in terms of clusters is:

$$|gs\rangle_4 = [\langle\uparrow\circ\downarrow\circ\rangle - \langle\downarrow\circ\uparrow\circ\rangle + \langle\circ\uparrow\circ\downarrow\rangle - \langle\circ\downarrow\circ\uparrow\rangle] \\ - [\langle 2\circ\circ\rangle + \langle\circ\, 2\circ\rangle + \langle\circ\circ 2\rangle + \langle 2\circ\circ\, 2\rangle], \tag{17}$$

where the notation $\langle 2\circ\circ 2\rangle$ indicates that the fermions at the border of the chain form a singlet (this is due to the periodic boundary conditions).

We have obtained further analytic results for chains of length $L = 5, 6, 7, 8$ with periodic boundary conditions. In particular we were interested in the eventual presence of supersymmetric ground states like the one appearing in the case $L = 4$. When $L = 5$ or $L = 7$, there is no state $|gs\rangle$ which satisfies $\overline{Q}_\alpha|gs\rangle = Q_\alpha|gs\rangle = 0$. In contrast, we have found such states in the cases $L = 6$ and $L = 8$. For $L = 6$, the supermultiplets formed by the $t = 1$ eigenvectors are shown in figure 10. We note the presence of a unique supersymmetric

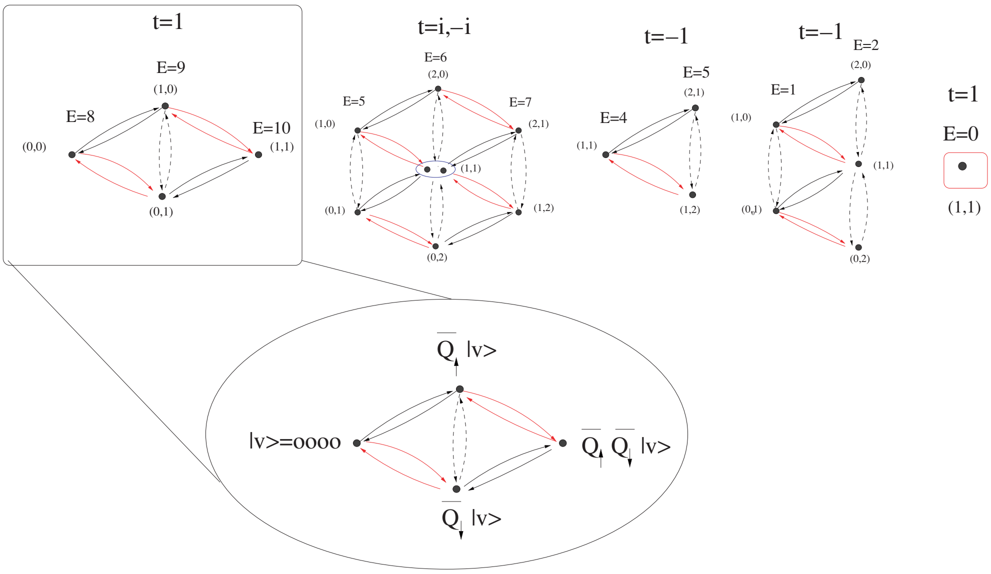

Figure 9. Spectrum of the model on a chain of length $L = 4$. The supermultiplet formed by acting with supersymmetry operators on the vacuum state is shown in detail.

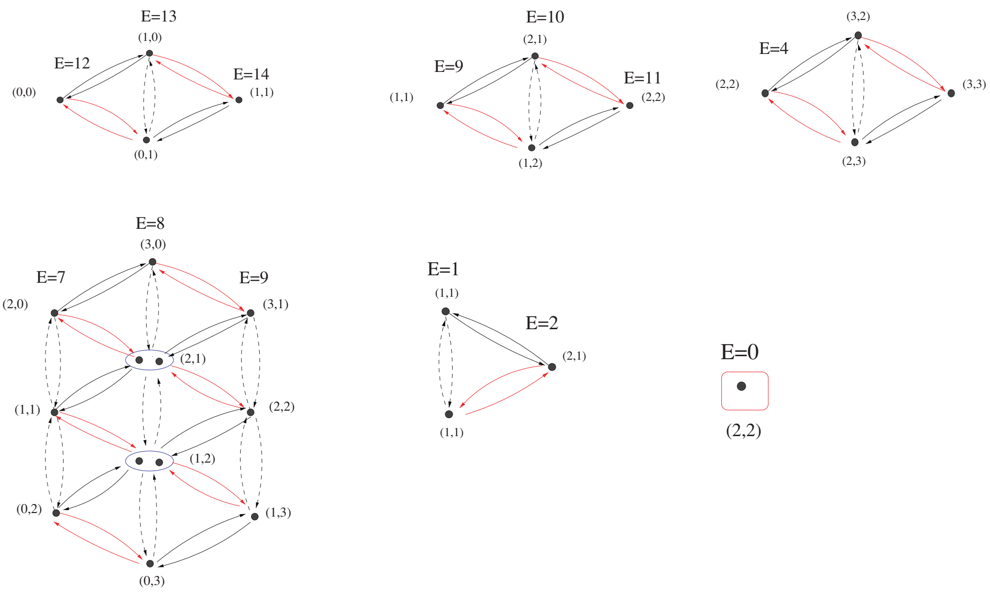

Figure 10. Supermultiplets formed by the translationally invariant eigenstates for a chain of length $L = 6$.

ground state $|gs\rangle_6$ in the sector $(1, 1)$, with the form

$$|gs\rangle_6 = 4[\langle\uparrow\circ\circ\downarrow\circ\circ\rangle - \langle\downarrow\circ\circ\uparrow\circ\circ\rangle] + 3[\langle\uparrow\circ\downarrow\circ\circ\circ\rangle - \langle\downarrow\circ\uparrow\circ\circ\circ\rangle] + [\langle 2\circ\circ\circ\circ\rangle] + \cdots, \tag{18}$$

with $\cdots$ representing all possible translations of the states shown. In the case $L = 8$, we have found a unique ground state $|gs\rangle_8$ in the sector $(2, 2)$ with translational eigenvalue $t = 1$. The corresponding expression is

$$|gs\rangle_8 = 4\sqrt{2}[\langle 3_\uparrow \circ\circ \downarrow \circ\circ\rangle - \langle 3_\downarrow \circ\circ \uparrow \circ\circ\rangle] \\ - 3\sqrt{2}[\langle 3_\uparrow \circ \downarrow \circ\circ\circ\rangle - \langle 3_\downarrow \circ \uparrow \circ\circ\circ\rangle] - 3\sqrt{2}[\langle \uparrow \circ 3_\downarrow \circ\circ\circ\rangle - \langle \downarrow \circ 3_\uparrow \circ\circ\circ\rangle] \\ + 4\langle 2\circ\circ 2\circ\circ\rangle + \langle 4\circ\circ\circ\circ\rangle + \cdots.$$

Again, $\cdots$ complete the expression to one that is translationally invariant.

Numerical diagonalization of the Hamiltonian up to $L = 14$ sites gave the following results. While for $L = 9$, $L = 11$ and $L = 13$ there are no supersymmetric ground states, such states do exist for $L = 10$, $L = 12$ and $L = 14$, in the sectors with fermion numbers (3, 3), (4, 4) and (5, 5), respectively.

At present, we do not have a good grasp of the pattern for the existence of supersymmetric ground states for general L. The quick arguments based on the Witten index, which guarantee the existence of such states in many $N = 2$ supersymmetric models, do not apply in these $N = 4$ supersymmetric models. From our explicit results up to $L = 14$, there is an obvious conjecture that supersymmetric ground states exist for general even L.

While, again, we lack a solid argument for determining the fermion number of these putative ground states, we observe the following. Focusing on a length-n string of consecutive fermions, one quickly finds that the potential energy $W_c(n)$ has a logarithmic dependence on n, $W_c(n) \sim \ln n$. In contrast, the contribution $W_h(n)$ to the potential energy from a string of n consecutive empty sites is linear in n, $W_h(n) \sim 2n$. Thus the system clearly favours the formation of large clusters. If indeed supersymmetric ground states exist for general even L, we expect them at a number of holes that either remains at a finite value or grows logarithmically with L (in this case the supersymmetric ground state will be in the sector (n, n) with $L - 2n \propto \ln L$).

Another important issue which we leave unsettled is that of conformal invariance. It will be interesting to determine if the model is critical, i.e., if the excitation energies above the ground state decrease as $1/L$ in the continuum limit. If this is the case, one expects that contact can be made with a form of conformal field theory.

3. Conclusion

In this concluding section we reiterate some of the remarkable properties of the model we considered, and we make some comments on related issues.

We have introduced a model of interacting spin-1/2 fermions on a chain with a manifest $SU(2)$ extended $N = 4$ supersymmetry. Our representation of $N = 4$ supersymmetry is highly nonlinear, as it is entirely built from degrees of freedom that are fermionic. We have looked for a supersymmetric model where $SU(2)$ spin symmetry is faithfully represented, and this has led us to a somewhat unusual restricted Hilbert space, with anti-ferromagnetic correlations built in from the start. The algebraic structure we have uncovered is very rich, but we are lacking a systematic mathematical framework. Such a framework will be most valuable, as it will allow us to further work out our present model and to decide on possibilities for alternative realizations of $N = 4$ supersymmetry.

We have found supersymmetric (zero energy) ground states for even L up to $L = 14$. In physical terms, they represent a (small) number of holes in an anti-ferromagnetic background. Our concrete realization of supersymmetry is restricted to one spatial dimension, but the general idea of exploiting supersymmetry is not. (See [5] where ground-state properties for spin-less fermions on a variety of $D = 2$ lattices are presented.) There is thus a possibility for exploiting supersymmetry in the analysis of doped antiferromagnets on $D = 2, 3$ lattices.

If this can be made to work, it is a potentially potent tool, which may supplement recent developments, where important progress on RVB states in $D = 2$ antiferromagnets was made with the help of the analysis of associated quantum dimer models [7].

Acknowledgments

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Appendix A. Amplitudes associated with the Hamiltonian

We give in this appendix the matrix elements defining the Hamiltonian (see section 2.4) in terms of the amplitudes of the supercharges.

Split–join terms

- $(2n)\circ(2m) \leftrightarrow (1-\delta_{\alpha,\beta})(2n+2m-2q-1)_\alpha \circ (2q+1)_\beta$, with $q = 0, 1, \ldots, n+m-1$:

$$\alpha,\beta = \uparrow,\downarrow \rightarrow \sum_\gamma z_{(2n+2m+1,2n+2m-2q),\gamma}\,\bar{y}_{(2n+2m+1,2n+1),\gamma} + \bar{y}_{(2n+2m-2q-1,2n+1),\uparrow}\,x_{(2m,2m-2q-1),\uparrow}$$

$$\alpha,\beta = \downarrow,\uparrow \rightarrow \sum_\gamma z'_{(2n+2m+1,2n+2m-2q),\gamma}\,\bar{y}_{(2n+2m+1,2n+1),\gamma} + \bar{y}_{(2n+2m-2q-1,2n+1),\downarrow}\,x_{(2m,2m-2q-1),\downarrow} \qquad \text{for } q = 0, 1, \ldots, m-1$$

and

$$\alpha,\beta = \uparrow,\downarrow \rightarrow \sum_\gamma z_{(2n+2m+1,2n+2m-2q),\gamma}\,\bar{y}_{(2n+2m+1,2n+1),\gamma} + \bar{y}_{(2m+2k+1,2k+1),\downarrow}\,x_{(2n,2n-2k),\downarrow}$$

$$\alpha,\beta = \downarrow,\uparrow \rightarrow \sum_\gamma z'_{(2n+2m+1,2n+2m-2q),\gamma}\,\bar{y}_{(2n+2m+1,2n+1),\gamma} + \bar{y}_{(2m+2k+1,2k+1),\uparrow}\,x_{(2n,2n-2k),\uparrow}$$

with $q = m, \ldots, m+n-1$ and $k = q-m$.

- $(2n)\circ(2m) \leftrightarrow (2n+2m-2q)\circ(2q)$, with $q = 0, 1, \ldots, m-1$:

$$\sum_\gamma [y_{(2n+2m+1,2n+2m-2q+1),\gamma}\,\bar{y}_{(2n+2m+1,2n+1),\gamma} + \bar{x}_{(2n+2m-2q,2n+1),\gamma}\,x_{(2m,2m-2q),\gamma}]$$

- $(2n)\circ(2m+1)_\alpha \leftrightarrow (2n+2m-2q+1)_\alpha \circ (2q)$ with $q = 0, 1, \ldots, n+m$:

$$(1-\delta_{\alpha,\beta})x_{(2n+2m+2,2n+2m-2q),\beta}\,\bar{x}_{(2n+2m+2,2n+1),\beta} + \bar{y}_{(2n+2m-2q+1,2n+1),\alpha}\,y_{(2m+1,2m-2q+1),\alpha} \qquad \text{for } q = 0, \ldots, m$$

and

$$(1-\delta_{\alpha,\beta})[x_{(2n+2m+2,2n+2m-2q),\beta}\,\bar{x}_{(2n+2m+2,2n+1),\beta} + \bar{x}_{(2m+2k,2k+1),\beta}\,x_{(2n,2n-2k),\beta}]$$

for $q = m+1, \ldots, n+m$ and $k = q-m-1$.

- $(2n)\circ(2m+1)_\alpha \leftrightarrow (2n+2m-2q)\circ(2q+1)_\alpha$ with $q = 0, \ldots, n+m-1$:

$$(1-\delta_{\alpha,\beta})[x_{(2n+2m+2,2n+2m-2q),\beta}\,\bar{x}_{(2n+2m+2,2n+1),\beta} + \bar{x}_{(2n+2m-2q,2n+1),\beta}\,t_{(2m+1,2m-2q),\beta}] + \bar{x}_{(2n+2m-2q,2n+1),\beta}\,z_{(2m+1,2m-2q),\alpha} \qquad \text{for } q = 0, \ldots, m-1$$

and

$$(1-\delta_{\alpha,\beta})[x_{(2n+2m+2,2n+2m-2q),\beta}\,\bar{x}_{(2n+2m+2,2n+1),\beta} + \bar{t}_{(2m+2k+1,2k),\beta}\,x_{(2n,2n-2k+1),\beta}] + \bar{z}_{(2m+2k+1,2k),\alpha}\,x_{(2n,2n-2k+1),\alpha}$$

for $q = m+1, \ldots, n+m$ and $k = q-m$.

Potential (diagonal) terms

- $(2n) \leftrightarrow (2n)$:

$$\sum_{\alpha} \sum_{q=1}^{2n} \bar{x}_{(2n,q),\alpha} x_{(2n,q),\alpha} + y_{(2n+1,1),\alpha} \bar{y}_{(2n+1,1),\alpha} + y_{(2n+1,2n+1),\alpha} \bar{y}_{(2n+1,2n+1),\alpha}$$

- $(2n+1)_\alpha \leftrightarrow (2n+1)_\alpha$:

$$\sum_{\alpha} \sum_{q=1,3,}^{2n+1} \bar{y}_{(2n+1,q),\alpha} y_{(2n+1,q),\alpha}$$

$$+ \sum_{\alpha} \sum_{q=2,4,}^{2n} [\bar{z}_{(2n+1,q),\alpha} z_{(2n+1,q),\alpha} + \bar{z}'_{(2n+1,q),\alpha} z'_{(2n+1,q),\alpha} + \bar{t}_{(2n+1,q),\alpha} t_{(2n+1,q),\alpha}]$$
$$+ x_{(2n+2,1),\alpha} \bar{x}_{(2n+2,1),\alpha} + x_{(2n+2,2n+2),\alpha} \bar{x}_{(2n+2,2n+2),\alpha}.$$

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